Approximate inertial manifolds for an atmospheric model of Lorenz

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Abstract. The paper establishes the existence of Approximate inertial manifolds for an atmospheric model of Lorenz together with local error estimates for the Nonlinear Galerkin approximation. Numerical studies are presented comparing this with the Linear Galerkin method.

Key words: Approximate inertial manifold; shallow water equations; Nonlinear Galerkin approximation.

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1 Introduction and preliminaries

The f-plane shallow-water equations under a time independent mass forcing $F(r)$, in which the horizontal and vertical motion of the fluid are diffusively damped, were introduced by Lorenz in 1980 [5]. These equations in non-dimensional form in the region $Q = [0, 2\pi] \times [0, 2\pi]$ are giving by:

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$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} + \vec{V}^{\perp} + \nabla z - \nu_0 \Delta \vec{V} = 0 \tag{1}$$

$$\frac{\partial z}{\partial t} + \nabla \cdot ((h_0 + z - \tilde{h})\vec{V}) - \kappa_0 \Delta z = F \tag{2}$$

Here, $\vec{V} = (u\ v)^T$ is the non-dimensional horizontal velocity of the fluid, and $\vec{V}^{\perp} = (-v\ u)^T$ is the orthogonal horizontal velocity. The deviation of the non-dimensional height of the fluid about its mean value h_0 is denoted by z and $\nu_0,\ \kappa_0$ corresponding to non-dimensional (renormalized) diffusivity coefficients. In what follows, we assume null adimensional topography ($\tilde{h} = 0$) and that $\kappa_0 = \nu_0$. For more details of these equations, see Lorenz [5], [6].

In Cárdenas and Thompson [2] the existence of the weak and strong solution to the parabolic equations (1) and (2) (without orography) for small initial data was established using Faedo-Galerkin method and uniqueness in the case of strong solutions. Moreover, in [2] an error estimate was determined for the Linear Galerkin (LG) approximation. Let us recall that if $p = P_n\xi$ is the aproximation Galerkin method over the exact solution ξ, then the residual $q = (I - P_n)\xi = \xi - p$ is not considered in the LG approximation. When the interactions between the components p and q are linked in the non-linear terms by the functional relation $q = \Phi(p)$, the graph (p, q) of the function Φ is called an Approximate Inertial Manifolds. These manifolds were introduced by Temam [9] in the context of the incompressivel Navier-Stokes equations. More refined algebraic and analytic approximations were subsequently introduced by Foias and Teman [3] based on the time analyticity of solutions on an attractor, but the analytic information available to us is weaker (under the hypotheses of Lemma 3.1), and we do not consider such approximations. In this paper, we establish the existence of Approximate inertial manifolds for the Atmospheric Lorenz system (1)-(2) (Sections 3 and 4) together with related error estimates (Theorem 4.1 and 4.2). We observe that these estimates are local in time for a spatial discretization of the system. The control on the spectrum of linerizations of the problem is insufficient to establish shadowing type estimates. Finally in section 5 we describe various numerical experiments for initial conditions under hydrostatic balance comparing the linear and non-linear Galerkin methods. We now introduce the requisite function spaces. Let

$$\mathcal{V} = \left\{ \xi = \begin{pmatrix} \vec{V} \\ z \end{pmatrix} \in \mathcal{P}, \text{ satisfying} \int_Q \vec{V}\, d\mathbf{x} = 0, \int_Q z\, d\mathbf{x} = 0 \right\}$$

where $\mathcal{P}$ is the trigonometric polynomials of period 2π in $\mathbf{x} = (x, y)$.

Let us define the spaces H=$L^2(Q)$ and V=$\dot{H}^1_{per}(Q)$ be the closure of $\mathcal{V}$ in $H^1(Q)$. The functions in H^m_{per} can be studied by means of Fourier series

$$\xi(\mathbf{x}, t) = \begin{pmatrix} \vec{V}(\mathbf{x}, t) \\ z(\mathbf{x}, t) \end{pmatrix} = \sum_{\mathbf{k} \in \mathcal{Z}^2} \begin{pmatrix} a_{\mathbf{k}}(t) \\ b_{\mathbf{k}}(t) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Thus $\xi \in H^m_{per}$ if and only if

$$|\vec{V}|^2_m = \sum_{\mathbf{k} \in \mathcal{Z}^2} |\mathbf{k}|^{2m} |\mathbf{a_k}|^2 < \infty \quad \text{and} \quad |z|^2_m = \sum_{\mathbf{k} \in \mathcal{Z}^2} |\mathbf{k}|^{2m} |b_{\mathbf{k}}|^2 < \infty.$$

Let us define the operator A_0 with $D(A_0) = \dot{H}^2_{per}(Q)$ by means of the bilinear form $(A_0\xi, \psi)_H = (\xi, \psi)_V$, $\xi,\ \psi \in$ V. A_0^{-1} is a linear continuous operator from H into $D(A_0)$ and compact, self-adjoint in H. Hence it possesses a sequence of eigenfunctions $w_j,\ j \in$ N, which forms an orthogonal basis of H such that

$$\begin{aligned} A_0 w_j = \lambda_j \omega_j, \quad \omega_j \in D(A_0), \\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to \infty \text{ for } j \to \infty. \end{aligned}$$

For velocities $\vec{U}$, $\vec{V}$ and geopotential z in V, the bilinear continuous forms $B(\vec{U}, \vec{V})$, $B_1(z, \vec{V})$ and $B_2(\vec{V}, z)$ from V×V in V* are defined by the following trilinear continuous forms

$$b(\vec{U}, \vec{V}, \vec{W}) = (B(\vec{U}, \vec{V})\ ,\ \vec{W}) = ((\vec{U} \cdot \nabla)\vec{V}\ ,\ \vec{W}) = \sum_{i,j=1}^{2} \int_Q u_i\ (D_i v_j)\ w_j\ dx,$$

$$b_1(z_1,\vec{V},z_2)=(B_1(z_1,\vec{V})\ ,\ z_2)=(z_1\nabla\cdot\vec{V}\ ,\ z_2)=\sum_{i=1}^{2}\int_Q z_1\ (D_iv_i)\ z_2\ dx,$$

$$b_2(\vec{V},z_1,z_2)=(B_2(\vec{V},z_1)\ ,\ z_2)=(\vec{V}\cdot\nabla z_1\ ,\ z_2)=\sum_{i=1}^{2}\int_Q v_i\ (D_iz_1)\ z_2\ dx.$$

We frequently make use of the following inequalities (see [8], [9] and [2]):

D1.- Poincare's inequality: $|\xi|_H \le \lambda_1^{-1/2}|\xi|_V, \quad \forall\xi\in$ V.

D2.- $|\xi|_V \le \lambda_1^{-1/2}|A_0\xi|_H, \quad \forall\xi\in D(A_0)$.

D3.- Young's inequality:

$$ab \le \frac{\epsilon}{p}a^p+\frac{1}{q\epsilon^{\frac{q}{p}}}b^q, \quad \forall a,b,\epsilon>0,\ \ 1<p<\infty,\ \ q=\frac{p}{p-1}.$$

E1.- $|b(\vec{U},\vec{V},\vec{W})|\ \le\ c_1|\vec{U}|_H^{\frac{1}{2}}|\vec{U}|_V^{\frac{1}{2}}\ |\vec{V}|_V|\vec{W}|_H^{\frac{1}{2}}|\vec{W}|_V^{\frac{1}{2}}, \quad \forall\ \vec{U},\vec{V},\vec{W}\in$ H

E2.- $|B(\vec{U},\vec{V})|_{V^*}\ \le\ c_2|\vec{U}|_H^{\frac{1}{2}}|A_0\vec{U}|_H^{\frac{1}{2}}|\vec{V}|_V, \quad \forall\ \vec{U}\in D(A_0),\ \vec{V}\in$ V.

E3.- $|b(\vec{U},\vec{V},\vec{W})|\le c_3|\vec{U}|_H|\vec{V}|_V^{\frac{1}{2}}|A_0\vec{V}|_H^{\frac{1}{2}}|\vec{W}|_H^{\frac{1}{2}}|\vec{W}|_V^{\frac{1}{2}},\ \forall\vec{U},\vec{W}\in$V, $\vec{V}\in D(A_0)$.

E4.- $|\int_D(\nabla\cdot\vec{U})\vec{V}\cdot\vec{W}|\le c_4|\vec{U}|_V^{\frac{1}{2}}|A_0\vec{U}|_H^{\frac{1}{2}}|\vec{V}|_H^{\frac{1}{2}}|\vec{V}|_V^{\frac{1}{2}}|\vec{W}|_H,\ \forall\vec{U}\in D(A_0),\ \vec{V},\vec{W}\in$V.

where d_i and c_i are absolute constants. Similar estimates are satisfied by the continuous form b_1, b_2, B_1 and B_2. In that follow, we denoting $|.|=|.|_H$, $\|.\|=\|.\|_V$. Therefore, $|\xi|_H^2=|\xi|^2=|\vec{V}|^2+|z|^2$ and $\|\xi\|_V^2=\|\xi\|^2=\|\vec{V}\|^2+\|z\|^2$.

2 The Approximate inertial manifold construction

Let P_m the L_2-projection on the first m-eigenfunctions of A_0 and let us set $\xi=\begin{pmatrix}\vec{V}\\ z\end{pmatrix}$, $\quad p=P_m\xi=\begin{pmatrix}P_m\vec{V}\\ P_mz\end{pmatrix}$, and $q=(I-P_m)\xi=Q_m\xi=\begin{pmatrix}Q_m\vec{V}\\ Q_mz\end{pmatrix}$, that is $\xi=p+q$. We call p by "the component of low modes" and q "the component of high modes". Decomposing p and q in the Fourier components of $\xi=p+q$, we obtaing the following norms and estimates,

$$|p|=\sqrt{\sum_{i=1}^{m}\xi_i^2};\quad \|p\|=\sqrt{\sum_{i=1}^{m}\lambda_i\ \xi_i^2};\quad |\Delta p|=\sqrt{\sum_{i=1}^{m}\lambda_i^2\ \xi_i^2};$$

$$|q| = \sqrt{\sum_{i=m+1}^{\infty} \xi_i^2}; \quad \|q\| = \sqrt{\sum_{i=m+1}^{\infty} \lambda_i\, \xi_i^2}; \quad |\Delta q| = \sqrt{\sum_{i=m+1}^{\infty} \lambda_i^2\, \xi_i^2};$$

$$|\Delta p| \leq \lambda_m^{1/2} \|p\| \ \leq \ \lambda_m |p|; \qquad |q| \ \leq \ \lambda_m^{-1/2} \|q\| \ \leq \ \lambda_m^{-1} |\Delta q|$$

where, $p = \sum_{i=1}^{m} \xi_i\ \omega_i$, $\quad q = \sum_{i=m+1}^{\infty} \xi_i\ \omega_i$, $\quad \xi_i = (\xi, \omega_i)$, and λ_i is the eigenvalue corresponding to ω_i eigenfunction of A_0 operator.

Using the projection operators P_m and Q_m em (1)-(2), we obtain the equivalent system of equations,

$$\frac{\partial p}{\partial t} = -\nu_0 A_0 p + P_m G(p+q) + P_m f, \qquad p(0) \ = \ P_m \xi_0. \tag{3}$$

$$\frac{\partial q}{\partial t} = -\nu_0 A_0 q + Q_m G(p+q) + Q_m f, \qquad q(0) \ = \ Q_m \xi_0. \tag{4}$$

with, $G(\xi) = \begin{pmatrix} -(\vec{V} \cdot \nabla)\vec{V} - \vec{V}^{\perp} - \nabla z \\ -\nabla \cdot ((h_0 + z)\vec{V}) \end{pmatrix}$, and $f = \begin{pmatrix} 0 \\ F \end{pmatrix}$.

We seek approximate solutions $\bar{\xi} = \bar{p} + \bar{q}$ for exact solution $\xi = p + q$ of the sistem (3)-(4). In particular if $\bar{q} = 0$, the dynamical system is approximated by the finite dynamical system,

$$\frac{\partial \bar{p}}{\partial t} = -\nu_0 A_0 \bar{p} + P_m G(\bar{p}) + P_m f, \qquad \bar{p}(0) \ = \ P_m \xi_0. \tag{5}$$

In this case, we have eliminated the interactions among the low and high modes, despising their contributions to the dynamics of the system. Error estimates for this approximation (Galerkin Methods) were established by Cárdenas and Thompson [2]. The graph $(\bar{p}, \bar{q})$ of Φ, is the Approximate inertial manifolds (AIM) for the Lorenz system (3)-(4), where the low mode component is determined by the equation

$$\frac{\partial \bar{p}}{\partial t} = -\nu_0 A_0 \bar{p} + P_m G(\bar{p} + \Phi(\bar{p})) + P_m f, \qquad \bar{p}(0) \ = \ P_m \xi_0, \tag{6}$$

and the high mode component $\bar{q}$ is determined as function of slow mode $\bar{p}$ by:

$$\bar{q} = \Phi(\bar{p}) \ = \ \nu_0^{-1} A_0^{-1} (Q_m G(\bar{p}) + Q_m f). \tag{7}$$

3 Preliminary estimates

As in the analysis of the paper by Heywood and Rannacher [4], it is necessary to establish the following Lemma:

Lemma 3.1 *Suppose that* $|A_0\xi(0)| < \infty$*, and*

$$\|\xi(t)\| \leq M_1, \qquad t \in [0, \infty) \tag{8}$$

then, there exits constants M_2 *and* M_3 *such that*

$$|A_0\xi(t)| \leq M_2, \quad |\xi_t(t)| \leq M_3, \qquad t > 0. \tag{9}$$

Proof. As to the proof we refer the reader to the results given in [2] (Lemma 4.1 and Lemma A.3). □

We choose $(p, \hat{q})$ to be another element of the graph Φ, where $\hat{q}$ is defined by:

$$\hat{q} = \Phi(p) \; = \; \nu_0^{-1} A_0^{-1}(Q_m G(p + Q_m f)). \tag{10}$$

The following estimate holds,

Lemma 3.2 *Under the hyphoteses of Lemma 3.1, we have*

$$|A_0(q - \hat{q})| \; \leq \; \nu_0^{-1} M_4 \, \|q\| + \nu_0^{-1} |q_t|$$

Proof. Let us substract (10) from (4) to obtain

$$\nu_0 A_0(q - \hat{q}) - Q(G(p + q) - G(p)) + q_t = 0.$$

It then follows that

$$\begin{aligned}
|A_0(q - \hat{q})| &\leq \nu_0^{-1}|Q_m G(p + q) - Q_m G(p)| + \nu_0^{-1}|q_t| \\
&\leq \nu_0^{-1}|q_t| + \nu_0^{-1}|Q_m \vec{V}^{\perp}| + \nu_0^{-1}\|Q_m z\| + g_0\nu_0^{-1}|\nabla \cdot Q_m \vec{V}| \\
&\quad + \nu_0^{-1}(|(P_m\vec{V} \cdot \nabla)Q_m\vec{V}| + |(Q_m\vec{V} \cdot \nabla)P_m\vec{V}| + |(Q_m\vec{V} \cdot \nabla)Q_m\vec{V}|) \\
&\quad + \nu_0^{-1}(|\nabla \cdot (P_m z Q_m \vec{V})| + |\nabla \cdot (Q_m z P_m \vec{V})| + |\nabla \cdot (Q_m z Q_m \vec{V})|.
\end{aligned}$$

The following bounds are established using the **E1**, **E2** and **E3**:

$$
\begin{aligned}
|Q_m\vec{V}^{\perp}| &\leq |Q_m\vec{V}| \;\leq\; \lambda_m^{-1/2}\|Q_m\vec{V}\| \;\leq\; \lambda_m^{-1/2}\|q\| \\
h_0|\nabla\cdot Q_m\vec{V}| &\leq \frac{3}{2}h_0\;\|Q_m\vec{V}\| \;\leq\; \frac{3}{2}h_0\;\|q\| \\
|(P_m\vec{V}\cdot\nabla)\;Q_m\vec{V}| &\leq c_2|P_m\vec{V}|^{1/2}\;|A_0\vec{V}|^{1/2}\;\|Q_m\vec{V}\| \\
&\leq c_2 M_2^{1/2}\lambda_1^{-1/4}\|P_m\vec{V}\|^{1/2}\|q\| \\
&\leq c_2\lambda_1^{-1/4}M_1^{1/2}M_2^{1/2}\|q\| \\
|(Q_m\vec{V}\cdot\nabla)P_m\vec{V}| &\leq c_3|Q_m\vec{V}|^{1/2}\;\|Q_m\vec{V}\|^{1/2}\;\|P_m\vec{V}\|^{1/2}|A_0\vec{V}|^{1/2} \\
&\leq c_3\lambda_m^{-1/4}\|Q_m\vec{V}\|^{1/2}\;\|P_m\vec{V}\|^{1/2}\lambda_m^{1/4}\;\|P_m\vec{V}\|^{1/2} \\
&\leq c_3 M_1\;\|q\| \\
|\nabla\cdot(P_m z\;Q_m\vec{V})| &\leq |P_m z\;\nabla\cdot Q_m\vec{V}| + |Q_m\vec{V}\cdot\nabla P_m z| \\
&\leq (c_2\lambda_1^{-1/4}M_1^{1/2}M_2^{1/2} + c_3 M_1)\;\|q\|
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
|(Q_m\vec{V}\cdot\nabla)\;Q_m\vec{V}| &\leq c_2\lambda_m^{-1/4}M_1^{1/2}M_2^{1/2}\|q\| \\
|\nabla\cdot(Q_m z P_m\vec{V})| &\leq (c_2\lambda_1^{-1/4}M_1^{1/2}M_2^{1/2} + c_3 M_1)\;\|q\| \\
|\nabla\cdot(Q_m z Q_m\vec{V})| &\leq 2c_2\lambda_1^{-1/4}M_1^{1/2}M_2^{1/2}\;\|q\|
\end{aligned}
$$

From the previous estimates we have that $|A_0(q-\hat{q})| \leq \nu_0^{-1}M_4\;\|q\| + \nu_0^{-1}|q_t|$, with $M_4 = 3c_2(\lambda_1^{-1/4}+\lambda_m^{-1/4})\;M_1^{1/2}M_2^{1/2} + c_3M_1) + \lambda_m^{-1/2} + \frac{3}{2}h_0$, and the proof is concluded. $\square$.

We may now establish one of the basic results of the AIM theory, Namely, the error between the exact value q and the approximation $\hat{q}$:

Theorem 3.1 *Under the hypotheses of the Lemma 3.1, the following error estimate for the high modes holds:*

$$|q-\hat{q}| \;\leq\; M_5\lambda_m^{-3/2}$$

Proof. We have

$$|q-\hat{q}| \leq \lambda_m^{-1}|A_0(q-\hat{q})| \;\leq\; \lambda_m^{-1}\nu_0^{-1}(M_4\|q\| + |q_t|), \quad \text{by Lemma 3.2.}$$

However, clearly we have

$$
\begin{aligned}
\|q\| &\leq \lambda_m^{-1/2}|A_0 q| \;\leq\; \lambda_m^{-1/2}M_2, \\
\text{and}\quad |q_t| &\leq \lambda_m^{-1/2}\|q_t\| \;\leq\; \lambda_m^{-1/2}M_3, \quad t>0.
\end{aligned}
$$

We conclude that

$$|q-\hat{q}| \leq \lambda_m^{-3/2}\nu_0^{-1}(M_4M_2+M_3) \ \leq \ \lambda_m^{-3/2}M_5,$$

and the result follows. □.

The following Lemma limits the distance in the norm of the gradient among two values any of the graph of Φ,

Lemma 3.3 *Let* (p_i,q_i), $i=1,2$, *belongs to graph of* Φ. *Then*

$$\|q_2-q_1\| \leq \nu_0^{-1}(M_6\|p_1\|+3c_1|p_2-p_1|)\ \|p_2-p_1\|.$$

Proof.

$$\begin{aligned}\nu_0\|q_2-q_1\|^2 &= (Q_m(G(p_2-p_1),q_2-q_1)\\ &= \Big(Q_m((P_m\vec{V}_2\cdot\nabla)P_m\vec{V}_2-(P_m\vec{V}_1\cdot\nabla)P_m\vec{V}_1),Q_m(\vec{V}_2-\vec{V}_1)\Big)\\ &\quad+\Big(Q_m(\nabla\cdot(P_mz_2\ P_m\vec{V}_2)-\nabla\cdot(P_mz_1\ P_m\vec{V}_1),Q_m(z_2-z_1)\Big).\end{aligned}$$

Observe that

$$\begin{aligned}\nabla\cdot(P_mz_2\ P_m\vec{V}_2-P_mz_1\ P_m\vec{V}_1) &= (P_mz_2\nabla\cdot P_m\vec{V}_2-P_mz_1\nabla\cdot P_m\vec{V}_1)\\ &\quad+(P_m\vec{V}_2\cdot\nabla P_mz_2-P_m\vec{V}_1\cdot\nabla P_mz_1)\end{aligned}$$

Then the quadratic terms are dominated using $E1$ by terms of the form

$$\begin{aligned}(p_2\cdot\nabla p_2-p_1\cdot\nabla p_1,q_2-q_1) &= (p_1\nabla\cdot(p_2-p_1)+(p_2-p_1)\nabla\cdot p_1\\ &\quad+(p_2-p_1)\nabla\cdot(p_2-p_1),q_2-q_1)\\ &\leq \ c_1(2\lambda_1^{-1/4}\|p_1\|+\lambda_m^{1/4}|p_2-p_1|)\ \|p_2-p_1\|\lambda_m^{-1/4}\|q_2-q_1\|\end{aligned}$$

Finally we obtain

$$\|q_2-q_1\| = \nu_0^{-1}(6c_1(\lambda_1\lambda_m)^{-1/4}\|p_1\|+3c_1|p_2-p_1|)\ \|p_2-p_1\|$$

and the proof is concluded taking $M_6=6c_1(\lambda_1\lambda_m)^{-1/4}$. □

Remark 3.1 *If in the Lemma 3.3 we take,* $p_1=p$, $q_1=\hat{q}$, $p_2=\bar{p}$, $q_2=\bar{q}$, *we have*

$$\|\bar{q}-\hat{q}\| \ \leq \ \nu_0^{-1}(M_6\|p\|+3c_1|\bar{p}-p|)\ \|\bar{p}-p\|$$

4 Error estimates for the low and high modes

The error estimate among $\bar{p}$ and p is established by the following theorem:

Theorem 4.1 *Under the hypotheses of Lemma 3.1, for a given $T > 0$, there exist m_*, M_7, depending on T, f, ν_0, and the given initial data, such that for every $m \geq m_*$, we have*

$$|\bar{p} - p| \leq M_7 \lambda_m^{-3/2}, \qquad t \in [0, T].$$

Proof. Let us first observe that from Lemma 3.1 and 3.2 we have the following estimate

$$\begin{aligned} |A_0(q - \hat{q}|) &\leq \nu_0^{-1}(M_4\|q\| + |q_t|) \ \leq \ \nu_0^{-1}(M_4\lambda_m^{-1/2}|A_0 q| + \lambda_m^{-1/2}\|q_t\|) \\ &\leq \nu_0^{-1}(M_4 M_2 + M_3)\lambda_m^{-1/2} \ \leq \ M_4' \lambda_m^{-1/2} \end{aligned} \tag{11}$$

Setting $\mathrm{w} = p - \bar{p} = \begin{pmatrix} \mathrm{w}_1 \\ \mathrm{w}_2 \end{pmatrix} = \begin{pmatrix} P_m(\vec{V} - \bar{\vec{V}}) \\ P_m(z - \bar{z}) \end{pmatrix}$, by Remark 3.1 we see that

$$\|\bar{q} - \hat{q}\| \leq \nu_0(M_6 M_1 + 3c_1|\mathrm{w}|)\ \|\mathrm{w}\| \tag{12}$$

Substracting (6) from (3) we have

$$\frac{d\mathrm{w}}{dt} + \nu_0 A_0 \mathrm{w} = P_m G(p + q) - P_m G(\bar{p} + \bar{q}).$$

Taking the inner product with w, we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathrm{w}|^2 + \nu_0\|\mathrm{w}\|^2 \leq |(P_m G(p + q) - P_m G(\bar{p} + \bar{q}), \mathrm{w})|. \tag{13}$$

The term $S = P_m(G(p + q) - G(\bar{p} + \bar{q}))$ can be written as,

$$S = P_m \begin{pmatrix} -B(\vec{V}, \vec{V}) + B(\bar{\vec{V}}, \bar{\vec{V}}) - \nabla \mathrm{w}_2 \\ -h_0 \nabla \cdot \mathrm{w}_1 - \nabla \cdot (z\vec{V}) + \nabla \cdot (\bar{z}\bar{\vec{V}}) \end{pmatrix}$$

The non-linear terms are decomposed as

$$\begin{aligned} -B(\vec{V}, \vec{V}) + B(\bar{\vec{V}}, \bar{\vec{V}}) &= -B(\vec{V}, \vec{V} - \bar{\vec{V}}) - B(\vec{V} - \bar{\vec{V}}, \vec{V}) + B(\vec{V} - \bar{\vec{V}}, \vec{V} - \bar{\vec{V}}) \\ &= -B(\vec{V}, \mathrm{w}_1) - B(\vec{V}, Q_m(\vec{V} - \bar{\vec{V}})) - B(\mathrm{w}_1, \vec{V}) \end{aligned}$$

$$-B(Q_m(\vec{V}-\vec{\bar{V}}),\vec{V}) + B(\mathrm{w}_1,\mathrm{w}_1) + B(\mathrm{w}_1,Q_m(\vec{V}-\vec{\bar{V}}))$$
$$+B(Q_m(\vec{V}-\vec{\bar{V}}),\mathrm{w}_1) + B(Q_m(\vec{V}-\vec{\bar{V}}),Q_m(\vec{V}-\vec{\bar{V}}))$$
$$-\nabla\cdot(z\vec{V}) + \nabla\cdot(z\vec{\bar{V}}) = B_1(z-\bar{z},\vec{V}-\vec{\bar{V}}) + B_2(\vec{V}-\vec{\bar{V}},z-\bar{z}) - B_1(z,\vec{V}-\vec{\bar{V}})$$
$$-B_2(\vec{V},z-\bar{z}) - B_2(\vec{V}-\vec{\bar{V}},z) - B_1(z-\bar{z},\vec{V})$$
$$= B_1(\mathrm{w}_2,\mathrm{w}_1) + B_1(\mathrm{w}_2,Q_m(\vec{V}-\vec{\bar{V}})) + B_1(z-\bar{z},\mathrm{w}_1) - B_1(z,\mathrm{w}_1)$$
$$+B_1(Q_m(z-\bar{z}),Q_m(\vec{V}-\vec{\bar{V}})) + B_2(\mathrm{w}_1,\mathrm{w}_2) + B_2(\mathrm{w}_1,Q_m(z-\bar{z}))$$
$$+B_2(Q_m(\vec{V}-\vec{\bar{V}}),\mathrm{w}_2) - B_1(z,Q_m(\vec{V}-\vec{\bar{V}})) - B_2(\vec{V},Q_m(z-\bar{z}))$$
$$+B_2(Q_m(\vec{V}-\vec{\bar{V}}),Q_m(z-\bar{z})) - B_2(\vec{V},\mathrm{w}_2) - B_2(Q_m(\vec{V}-\vec{\bar{V}}),z)$$
$$-B_2(\mathrm{w}_1,z) - B_1(\mathrm{w}_2,\vec{V}) - B_1(Q_m(z-\bar{z}),\vec{V})$$

The non-linear terms are estimated using the Sobolev inequalities D1-D3, E1-E4, the equality $q-\bar{q}=(q-\hat{q})+(\hat{q}-\bar{q})$ and the estimates (11)-(12), in the following sequence of estimates (a) to (i):

(a) $$|b(\vec{V},\mathrm{w}_1,\mathrm{w}_1)| \le c_2|\vec{V}|^{1/2}|A_0\vec{V}|^{1/2}\|\mathrm{w}_1\|\,|\mathrm{w}_1| \le c_2 M_1^{1/2}M_2^{1/2}\|\mathrm{w}\|\,|\mathrm{w}| \le \frac{\epsilon}{2}\|\mathrm{w}\|^2 + \frac{1}{2\epsilon}c_2^2 M_1 M_2 |\mathrm{w}|^2$$

(b) $$|b(\vec{V},Q_m(\vec{V}-\vec{\hat{V}}),\mathrm{w}_1)| \le |B(\vec{V},\mathrm{w}_1),Q_m(\vec{V}-\vec{\hat{V}}))| + |\int_D(\nabla\cdot\vec{V})\mathrm{w}_1 Q_m(\vec{V}-\vec{\hat{V}}))|$$
$$\le (c_2+c_4)|A_0\vec{V}|^{1/2}|\,|q-\hat{q}|\,(|\vec{V}|^{1/2}\|\mathrm{w}\| + \|\vec{V}\|^{1/2}|\mathrm{w}|^{1/2}\|\mathrm{w}\|^{1/2})$$
$$\le (c_2+c_4)\lambda_1^{-1/4}M_1^{1/2}M_2^{1/2}M_4'\lambda_m^{-3/2}\|\mathrm{w}\|$$
$$\le \frac{\epsilon}{2}\|\mathrm{w}\|^2 + \frac{1}{2\epsilon}(c_2+c_4)^2 M_1 M_2 M_4'^2\lambda_m^{-3}$$

(c) $$|b(\vec{V},Q_m(\vec{\bar{V}}-\vec{\hat{V}}),\mathrm{w}_1)| \le |B(\vec{V},\mathrm{w}_1),Q_m(\vec{\bar{V}}-\vec{\hat{V}}))| + |\int_D(\nabla\cdot\vec{V})\mathrm{w}_1 Q_m(\vec{\bar{V}}-\vec{\hat{V}}))|$$
$$\le (c_2+c_4)|A_0\vec{V}|^{1/2}|\,|\bar{q}-\hat{q}|\,(|\vec{V}|^{1/2}\|\mathrm{w}\| + \|\vec{V}\|^{1/2}|\mathrm{w}|^{1/2}\|\mathrm{w}\|^{1/2})$$
$$\le (c_2+c_4)\lambda_1^{-1/4}M_1^{1/2}M_2^{1/2}\lambda_m^{-1/2}\|\bar{q}-\hat{q}\|\,\|\mathrm{w}\|$$
$$\le (c_2+c_4)\lambda_1^{-1/4}M_1^{1/2}M_2^{1/2}\lambda_m^{-1/2}\nu_0^{-1}(M_6M_1+3c_1|\mathrm{w}|)\,\|\mathrm{w}\|^2$$

(d) $$|b(\mathrm{w}_1,\vec{V},\mathrm{w}_1)| \le c_1|\mathrm{w}_1|^{1/2}\|\mathrm{w}_1\|^{1/2}\|\vec{V}\|\,|\mathrm{w}_1|^{1/2}\|\mathrm{w}_1\|^{1/2} \le c_1M_1|\mathrm{w}|\,\|\mathrm{w}\| \le \frac{\epsilon}{2}\|\mathrm{w}\|^2 + \frac{1}{2\epsilon}c_1^2M_1^2\,|\mathrm{w}|^2$$

(e) $$|b(Q_m(\vec{\bar{V}}-\vec{\hat{V}}),\vec{V},\mathrm{w}_1)| \le c_3\,|q-\hat{q}|\,\|\vec{V}\|^{1/2}|A_0\vec{V}|^{1/2}|\mathrm{w}|^{1/2}\,\|\mathrm{w}\|^{1/2}$$

$$\leq c_3 M_1^{1/2} M_2^{1/2} \lambda_1^{-1/4} \lambda_m^{-1} |A_0(q-\hat{q})| \; \|\mathrm{w}\|$$
$$\leq c_3 M_1^{1/2} M_2^{1/2} M_4' \lambda_1^{-1/4} \lambda_m^{-3/2} \|\mathrm{w}\|$$
$$\leq \frac{\epsilon}{2}\|\mathrm{w}\|^2 + \frac{1}{2\epsilon} c_3^2 M_1 M_2 M_4'^2 \lambda_1^{-1/2} \lambda_m^{-3}$$

(f) $$|b(Q_m(\vec{\bar{\bar{V}}} - \vec{\hat{V}}), \vec{V}, \mathrm{w}_1)| \leq c_3 \; |\bar{q}-\hat{q}| \; \|\vec{V}\|^{1/2} |A_0 \vec{V}|^{1/2} |\mathrm{w}|^{1/2} \; \|\mathrm{w}\|^{1/2}$$
$$\leq c_3 M_1^{1/2} M_2^{1/2} \lambda_1^{-1/4} \lambda_m^{-1/2} \|\bar{q}-\hat{q}\| \; \|\mathrm{w}\|$$
$$\leq \; c_3 M_1^{1/2} M_2^{1/2} \lambda_1^{-1/4} \lambda_m^{-1/2} \nu_0^{-1} \; (M_6 M_1 + 3c_1 |\mathrm{w}|) \; \|\mathrm{w}\|$$

(g) $$|b(\mathrm{w}_1, \mathrm{w}_1, \mathrm{w}_1)| \leq c_1 |\mathrm{w}|^{1/2} \|\mathrm{w}\|^{1/2} \|\mathrm{w}\| \; |\mathrm{w}|^{1/2} \|\mathrm{w}\|^{1/2} \; \leq \; c_1 |\mathrm{w}| \; \|\mathrm{w}\|^2$$

(h) $$|b(Q_m(\vec{\bar{V}} - \vec{\hat{V}}), \mathrm{w}_1, \mathrm{w}_1)| \leq c_1 \lambda_m^{-1/4} \|\bar{q}-\hat{q}\| \; \|\mathrm{w}\| \; |\mathrm{w}|^{1/2}| \; \|\mathrm{w}\|^{1/2}$$
$$\leq c_1 \lambda_m^{-1/4} \nu_0^{-1} (M_6 M_1 + 3c_1|\mathrm{w}|) \; \lambda_m^{1/4} |\mathrm{w}| \; \|\mathrm{w}\|^2$$
$$\leq c_1 \nu_0^{-1} (M_6 M_1 |\mathrm{w}| + 3c_1 |\mathrm{w}|^2) \; \|\mathrm{w}\|^2$$

(i) $$|b(Q_m(\vec{\bar{V}} - \vec{\hat{V}}), Q_m(\vec{\bar{V}} - \vec{\hat{V}}), \mathrm{w}_1)| \leq c_1 \lambda_m^{-1/4} \; \|\bar{q}-\hat{q}\|^2 \lambda_m^{1/4} |\mathrm{w}|$$
$$\leq 2c_1 \nu_0^{-2} (M_6^2 M_1^2 |\mathrm{w}| + 9c_1^2 |\mathrm{w}|^3) \; \|\mathrm{w}\|^2.$$

The others terms are estimated in a similar way. Taking $\epsilon = \nu_0/14$, we obtain a term of the form $\nu_0 \|\mathrm{w}\|^2/2$ which is absorbed in (13). The final estimate is of the type

$$\frac{d}{dt}|\mathrm{w}|^2 + \nu_0 \|\mathrm{w}\|^2 \leq \alpha |\mathrm{w}|^2 + (r_1 \lambda_m^{-1/2} + r_2 |\mathrm{w}| + r_3 |\mathrm{w}|^2 + r_4 |\mathrm{w}|^3) \; \|\mathrm{w}\|^2 + s\lambda_m^{-3} \tag{14}$$

which can be analysed by mean of the following argument. First note that, for $T > 0$ fixed, considering the differential inequality (14) we have

$$\frac{d}{dt}|\mathrm{w}|^2 \leq \alpha |\mathrm{w}|^2 + s\lambda_m^{-3}, \qquad \text{then} \quad |\mathrm{w}|^2 \; \leq \; \frac{s\lambda_m^{-3}}{\alpha}(e^\alpha - 1) \; \leq \; K(T)\lambda_m^{-3}.$$

Choosing m^* such that, $K(T)\lambda_m^{-3} \leq 1$, $r_1 \lambda_m^{-1/2} \leq \nu_0/8$, $\max\{r_2, r_3, r_4\} \leq \nu_0/8$, the non-linear terms are absorbed in the left hand, so that

$$\frac{d}{dt}|\mathrm{w}|^2 + \frac{1}{2}\nu_0 \|\mathrm{w}\|^2 \leq \alpha |\mathrm{w}|^2 + s\lambda_m^{-3}$$

The integrating on the interval $[0,T]$ results in $|\mathrm{w}|^2 \;\leq\; K(T)\lambda_m^{-3}$. Hence, $|\mathrm{w}| \;\leq\; K^{1/2}(T)\lambda_m^{-3/2}$, and the proof is conclude taking $M_7 = K^{1/2}(T)$. □

The error estimate for the difference between $\bar{q}$ and q is established by the following theorem:

Theorem 4.2 *Under the hypotheses of the Theorem 4.1, one has the estimate:*

$$|\bar{q} - q| \;\leq\; M_8\lambda_m^{-3/2}, \qquad t \in [0,T].$$

Proof. By (11) we have $|q-\hat{q}| \leq \lambda_m^{-1}|A_0(q-\hat{q})| \leq M_4^{'}\lambda_m^{3/2}$. On the other hand from (12) and Theorem 4.1 we obtain that

$$\begin{aligned} |\bar{q} - \hat{q}| &\leq \lambda_m^{-1/2}\|\bar{q} - \hat{q}\| \;\leq\; \lambda_m^{-1/2}\nu_0^{-1}(M_6M_1 + 3c_1|\mathrm{w}|)\;\|\mathrm{w}\| \\ &\leq \lambda_m^{-1/2}\nu_0^{-1}(M_6M_1 + 3c_1M_8\lambda_m^{-3/2})\;\lambda_m^{1/2}M_8\lambda_m^{-3/2} \\ &\leq \nu_0^{-1}(M_6M_1 + 3c_1M_7\lambda_m^{-3/2})\;M_7\lambda_m^{-3/2} \end{aligned}$$

Then,

$$|q - \bar{q}| \leq |q - \hat{q}| + |\bar{q} - \hat{q}| \;\leq\; M_8\lambda_m^{-3/2},$$

with, $M_8 = M_4^{'} + \nu_0^{-1}(3M_6M_1c_1M_7\lambda_m^{-3/2})M_7$, and the proof is concluded. □

Of the Theorems 4.1 and 4.2 we have the error estimate for the difference between $\xi = p + q$ and $\bar{\xi} = \bar{p} + \bar{q}$:

Corollary 4.1 *Let $\xi = p+q$ be solution of the system (3)-(4), and $\bar{\xi} = \bar{p}+\bar{q}$ be solution of the system (6)-(7), then one has the following error estimate:*

$$|\bar{\xi} - \xi| \;=\; |(p+q) - (\bar{p}+\bar{q}| \;\leq\; |\bar{p}-p| + |\bar{q}-q| \;\leq\; (M_7+M_8)\lambda_m^{-3/2}.$$

5 Numerical experiments

We consider a second-order semi-implicit time discretization of (1)-(2) equations, where the terms giving rise to the fast gravity waves (such as the geopotential gradient) will be treated implicity, while the nonlinear term will be discretized explicitly by a leap-frog type scheme. Similar scheme discretization is applied for the slow frequencies equation (6), where is used the approximate inertial manifold (7) in the non-linear terms. We refer the reader to the paper by Barros and Cardenas [1] for detailed numerical implementation of these equations. We have realized experiments with and without use of the approximate inertial manifold $q = \Phi(p)$ for several initial conditions and calculated the mean error over the domain for each time step integration. In the Figure 1 there is shown the evolutions of error for initial conditions in hydrostatic balance and given force term $F(x, y)$ in the height (geopotential) equation. For evaluation of the error, a comparision solution is considered using integration with 128 modes. Then, the equations (1)-(2) (the so called linear Gakerkin method (LG)) are integrated with 16 and 32 modes. On the other hand, the equation (6) is integrated with 16 modes where there is used the approximate inertial manifold (7) (the so called non-linear Galerkin method (NLG)). The results show that the use of diagnostic equations (7) permits diminished the error if compared with the system (1)-(2) with the same number modes. Effectively, the error of the velocities and the height geopotencial for the NLG method with 16 modes is indistinguishable from the error of the LG method with 32 modes, while both errors are smaller that the LG method with 16 modes.

Figure 2 is shows the quocient between the error of the several experiments

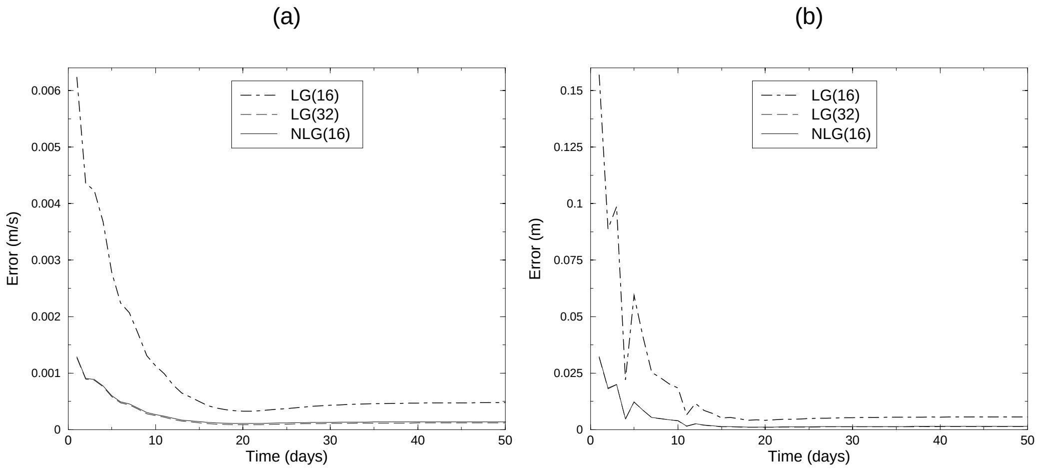

Fig. 1. Temporal series of the spatial error: a) zonal velocity (m/s) and b) height geopotential (m).

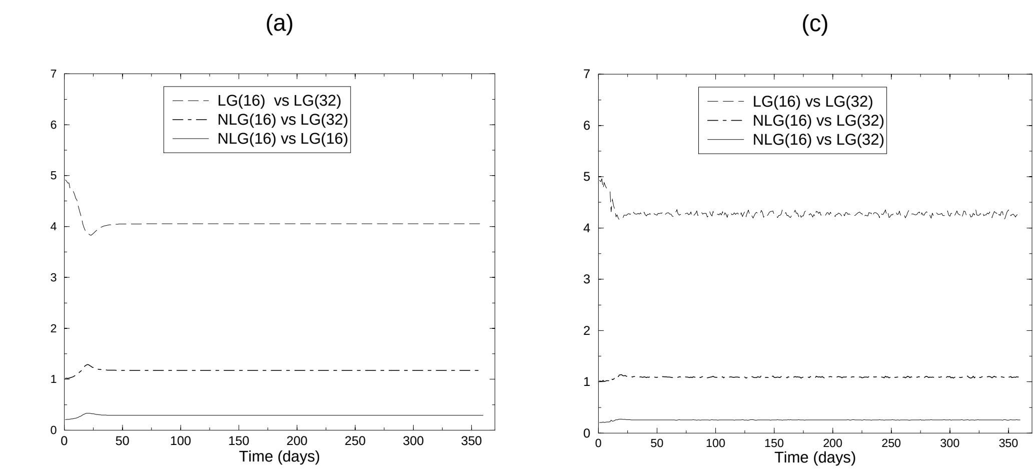

Fig. 2. Quotients of the errors: a) zonal velocity and b) height geopotential.

to the velocity and geopotential height. One observes a factor 4 between the errors of the LG with 16 modes divided by LG with 32 modes, a factor 1 between the errors of the NLG with 16 modes divided by LG com 32 modes, and finally a factor of 0.25 between the errors of the NLG with 32 modes by LG with 16 modes. These numerical results are compatible with the following estimates,

$$\frac{|\xi_m^{LG} - \xi|}{|\xi_{2m}^{LG} - \xi|} \sim \frac{c_1 \lambda_m^{-1}}{c_1 \lambda_{2m}^{-1}} \sim \frac{m^{-2}}{(2m)^{-2}} \sim 4$$

$$\frac{|\xi_m^{NLG}-\xi|}{|\xi_{2m}^{LG}-\xi|} \sim \frac{c_2\lambda_m^{-3/2}}{c_1\lambda_{2m}^{-1}} \sim \frac{c_2 m^{-3}}{c_1(2m)^{-2}} \sim \frac{4c_2}{c_1\ m} \sim 1, \quad \text{if} \quad c_2 = \frac{c_1 m}{4}$$
$$\frac{|\xi_m^{NLG}-\xi|}{|\xi_{m}^{LG}-\xi|} \sim \frac{c_2\lambda_m^{-3/2}}{c_1\lambda_{m}^{-1}} \sim \frac{c_2 m^{-3}}{c_1(m)^{-2}} \sim \frac{c_2}{c_1\ m} \sim \frac{1}{4}, \quad \text{if} \quad c_2 = \frac{c_1 m}{4}$$

where, ξ_m^{LG}, ξ_m^{LNG} are the LG and NLG approximate solution using m modes respectively, and ξ is the exact solution. However, for $m > m^*$, the errors satisfing $|\xi_m^{LG} - \xi| \le c_1\lambda_m^{-1}$ (see [2]), and $|\xi_m^{LNG} - \xi| \le c_2\lambda_m^{-3/2}$ (see Corollary 4.1). Therefore, if $c_2 = c_1 m/4$ the NLG error numerical estimates verify the estimative, $|\xi_m^{LNG} - \xi| \le c_1\lambda_m^{-1}/4$. That is, 25% less than error of the LG method with m modes, and the similar error that LG method with $2m$ modes. Several numerical experiments with different initial conditions and resolutions have shown the same behaviour as that of the Figure 1 and 2.

6 Concluding remarks

In [2] the error of the Linear Galerkin method for an atmospheric model of Lorenz was estimated. This error was of the order λ_m^{-1}, larger than the error of the order $\lambda_m^{-3/2}$ obtained here with the use of the Approximate inertial manifold. The action of the high modes $q = \Phi(p)$ in the non linear terms contribute to diminish the error, this is verified in numerical experiments. The present paper provides an analytic basis for the semi discretizations for a nonlinear Galerkin methods. Fully discretized numerical methods based in these manifolds (the so-called Nonlinear Galerkin methods [7]) have been applied for the atmospheric model of Lorenz (1)-(2) by Barros and Cárdenas [1]. In future work we propose to study approximations of slow manifolds associated with the balanced atmospheric model of Lorenz with relatively small Rossby number.

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